

Numerical Simulation of the Buckley-Leverett Problem

by

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Abstract

The displacement of oil by water in a petroleum reservoir is analyzed via the famous Buckley-Leverett displacement theory. A quantitative demonstration is performed for a typical relative permeability data and reservoir condition in reference to the literature by Aziz and Settari. The immiscible displacement process is then analyzed numerically by the finite difference method and the finite element method, in which the flow equations of the two-phase liquid are transformed into a single-valued pressure equation, i.e. capillary pressure is neglected, and water saturation equation. The pressure equation is solved implicitly, and the saturation equation is solved explicitly with a special attention to the time step level on the pressure-saturation evaluation. Numerical simulations developed in this paper show good correlation with the results by the Buckley-Leverett analysis.

Keywords: Petroleum Reservoir Engineering, Waterflooding, Two-phase Flow, Immiscible Displacement, Buckley-Leverett Theory, Finite Difference Method, Finite Element Method

1. Introduction

In petroleum reservoir engineering, a technique of injecting water into reservoir has been used in order to maintain oil production rates during the pumping operation. The method is known as the waterflooding technique, which provides high oil production rates and high degree of petroleum recovery when oil production rates deteriorate¹⁾.

When water is injected into reservoir, oil is displaced toward the production well in the situation of two-phase flow as illustrated in Fig.1. Oil and water are immiscible in each other, so that this phenomenon is referred to the immiscible displacement in porous media. The mechanism of immiscible displacements of two-phase fluids was studied extensively and a number of technical papers have been published. The special case of one-dimensional, incompressible, two-phase flow was theoretically investigated first by Buckley and Leverett²⁾ in 1942.

The Buckley and Leverett frontal displacement theory described a method for calculating saturation profiles when the effects of capillary pressure and gravity are neglected. However, the theory gained popularity in the petroleum engineering because of its simplicity and accuracy¹⁾.

Since appearance of the theory, numerical analysis was eagerly attempted to simulate the Buckley-Leverett problem.

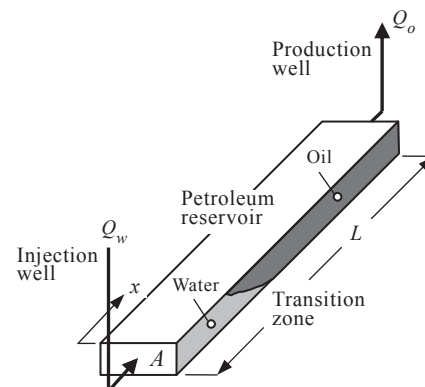


Fig.1 Waterflooding method for oil production.

The early numerical methods were simultaneous solution method (SS) and implicit pressure-explicit saturation solution (IMPES) by Finite Difference Method³⁾. The analysis was then extended to the case which includes the effect of gravity, capillary pressure, fluid compressibility and aquifer compressibility. Through the numerical analyses by FDM, it was found that, as time progress, the saturation becomes a multiple-valued function of the grid spacing⁴⁾. The Finite Element Method was also developed to the analysis of flow through porous media, and applied to the two-phase flow problems. There are common numerical difficulties in FDM and FEM that reverse changes in mobility factors in water and oil phases appear in the numerical computation process, which results in a convection-dominated situation⁵⁾. The

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upstream weighting for mobility factors is required to avoid these difficulties in both FDM and FEM^{6,7)}.

In this paper, Buckley-Leverett analysis is performed first based on the relative permeability data employed by Azis and Settari⁴⁾, and the finite difference and the finite element analysis are attempted using the same physical data adopted in the Buckley-Leverett analysis. The numerical analysis is basically based on the method of IMPES, but special attention has been paid to the time step level in the pressure change in the numerical formulation.

2. Buckley-Leverett Analysis

In the first, the Buckley-Liverett frontal displacement theory is reviewed, and an example of the analysis is presented. The flow rate of oil and water through completely saturated porous medium in horizontal direction is given by the Darcy's law as follows.

$$Q_o = -\frac{k_x k_{ro} A}{\mu_o} \frac{\partial p_o}{\partial x}, \quad Q_w = -\frac{k_x k_{rw} A}{\mu_w} \frac{\partial p_w}{\partial x} \quad (1)(2)$$

where k_x is the intrinsic permeability of the medium, A is the cross-sectional area for permeation, μ_o and μ_w are the dynamic viscosity of oil and water, p_o and p_w are the pore pressure of oil and water, and k_{ro} and k_{rw} are the relative permeability of oil and water respectively. The relative permeability k_{ro} and k_{rw} are generally given as a function of water saturation S_w as shown in Fig.2, where S_{wr} and S_{or} are the residual water and oil saturation. The pressure difference at the contact surface between oil and water, i.e. capillary pressure, is denoted by $p_{o/w}$, then p_w of Eq.(2) is rewritten as

$$Q_w = -\frac{k_x k_{rw} A}{\mu_w} \frac{\partial (p_o - p_{o/w})}{\partial x} \quad (3)$$

and subtracting Eq.(1) from Eq.(3) yields

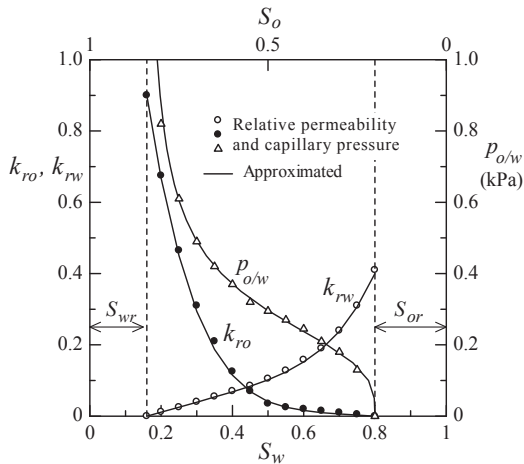


Fig.2 Relative permeabilities and capillary pressure with saturation⁴⁾.

$$-\frac{1}{k_x A} (Q_w \frac{\mu_w}{k_{rw}} - Q_o \frac{\mu_o}{k_{ro}}) = -\frac{\partial p_{o/w}}{\partial x} \quad (4)$$

From the relation of $Q_o + Q_w = Q_T$, substitution of $Q_o = Q_T - Q_w$ into Eq.(4) yields

$$Q_T = Q_w (1 + \frac{k_{ro}}{k_{rw}} \frac{\mu_w}{\mu_o}) - \frac{k_x k_{ro} A}{\mu_o} \frac{\partial p_{o/w}}{\partial x} \quad (5)$$

from which the fraction of pore water flow, i.e., fractional flow rate f_w , is expressed as

$$f_w = \frac{Q_w}{Q_T} = \frac{1 + \frac{k_x k_{ro} A}{Q_T \mu_o} \frac{\partial p_{o/w}}{\partial x}}{1 + \frac{k_{ro}}{k_{rw}} \frac{\mu_w}{\mu_o}} \quad (6)$$

The capillary pressure $p_{o/w}$ is actually very small compared to the water pressure when water displaces oil in the reservoir. If the effect of capillary pressure is neglected, Eq.(6) becomes the following simple expression.

$$f_w = \frac{1}{1 + \frac{k_{ro}}{k_{rw}} \frac{\mu_w}{\mu_o}} \quad (7)$$

Next, continuity equation of pore water is expressed as

$$-\frac{\partial Q_w}{\partial x} = A \phi \frac{\partial S_w}{\partial t} \quad (8)$$

where ϕ is the porosity of the porous medium. Using the relations of $Q_w = f_w Q_T$ and $f_w(S_w)$, Eq.(8) is rewritten as

$$\left(\frac{\partial S_w}{\partial t}\right)_x = -\frac{Q_T}{A \phi} \left(\frac{df_w}{dS_w} \frac{\partial S_w}{\partial x}\right)_t \quad (9)$$

Equation (9) is a first-order nonlinear hyperbolic equation. The main and very fruitful idea of Buckley and Leverett was to transform Eq.(9) into the following form:

$$\left(\frac{\partial x}{\partial t}\right)_{S_w} = \frac{Q_T}{A \phi} \left(\frac{df_w}{dS_w}\right)_t \quad (10)$$

stating that the rate of advance of a plane of fixed saturation S_w is proportional to the rate of change in composition of the flowing stream with saturation. As f_w is not an explicit function of t , Eq.(10) can be integrated to give the position of a particular saturation as a function of time:

$$x_{S_w} = \frac{Q_T t}{A \phi} \left(\frac{df_w}{dS_w}\right) + x_0 \quad (11)$$

where x_0 is the position of the water saturation at time $t=0$.

According to Eq.(11), each saturation advances into the system at a rate in direct proportion to $f_w' = df_w/dS_w$. The shape of the saturation profile calculated by Eq.(11) is exp-

3.1 Finite Difference Method

The finite difference method has a long history and has been applied to various problems of flow through porous media. The equations of flow for water and oil in an isotropic-homogenous incompressible medium are written as

$$\frac{\partial}{\partial x} (k_x \lambda_w \frac{\partial p_w}{\partial x}) = \phi \frac{\partial S_w}{\partial t} \quad (12)$$

$$\frac{\partial}{\partial x} (k_x \lambda_o \frac{\partial p_o}{\partial x}) = \phi \frac{\partial S_o}{\partial t} \quad (13)$$

where $\lambda_w = k_{rw}/\mu_w$ and $\lambda_o = k_{ro}/\mu_o$ are mobility factors, and p_w and p_o are fluid pressures in each phase. There exists the following axially relationships for saturation and pressure;

$$S_w + S_o = 1 \quad , \quad p_o - p_w = p_{o/w} \quad (14a, b)$$

If the capillary pressure $p_{o/w}$ is neglected, i.e. $p_o = p_w$, the water and oil pressures are replaced by a single variable p , and Eqs.(12)(13) are rewritten as

$$\frac{\partial}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x}) = \phi \frac{\partial S_w}{\partial t} \quad (15)$$

$$\frac{\partial}{\partial x} (k_x \lambda_o \frac{\partial p}{\partial x}) = -\phi \frac{\partial S_w}{\partial t} \quad (16)$$

By adding the water and oil phase equations of Eq.(15) and Eq.(16), the equation of two-phase flow is described as

$$\frac{\partial}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x}) + \frac{\partial}{\partial x} (k_x \lambda_o \frac{\partial p}{\partial x}) = 0 \quad (17)$$

Now, time derivative term of saturation has been eliminated, Eq.(17) is called the pressure equation. It should be emphasized that Eq.(17) is not a steady state equation but a transient equation with respect to pressure change, because the mobility factors undergo the change of water saturation. The boundary conditions may be given by

$$-k_x \lambda_w \frac{\partial p}{\partial x} = \frac{\hat{Q}_r}{A} \quad \text{at } x=0 \quad , \quad p = \hat{p} \quad \text{at } x=L \quad (18a, b)$$

where \hat{p} is the prescribed pressure at the outlet. Hereupon, the pressure equation (17) is transformed to the finite difference equation, and the resulting algebraic equation can be written in general form as

$$-A_i p_{i+1}^{k+\theta} + B_i p_i^{k+\theta} - C_i p_{i-1}^{k+\theta} = D_i \quad (19)$$

If θ is chosen to be 1, Eq.(19) becomes fully implicit scheme. In this study, the time step level for the computation of p was selected at $k+1/2$ because pressure change in flow equations (15)(16) should be evaluated between k and $k+1$. Thus,

$$-A_i p_{i+1}^{k+1/2} + B_i p_i^{k+1/2} - C_i p_{i-1}^{k+1/2} = D_i \quad (20)$$

where

$$\begin{aligned} A_i &= \gamma (\lambda_{wi} + \lambda_{oi}) \quad , \quad B_i = A_i + C_i \\ C_i &= \gamma (\lambda_{wi-1} + \lambda_{oi-1}) \quad , \quad D_i = 0 \end{aligned} \quad (21a, b, c, d)$$

in which $\gamma = k_x \Delta t / \Delta x^2$. The mobility factors $\lambda_{o/w}$ are normally evaluated at the upstream side nodes as shown in Fig.6, i.e. upstream weighting, to obtain stable solutions. If $\lambda_{o/w}$ are taken at the intermediate points, i.e. midstream weighting, the coefficients in Eq.(20) are written as

$$\begin{aligned} A_i &= \gamma (\lambda_{wi+1/2} + \lambda_{oi+1/2}) \quad , \quad B_i = A_i + C_i \\ C_i &= \gamma (\lambda_{wi-1/2} + \lambda_{oi-1/2}) \quad , \quad D_i = 0 \end{aligned} \quad (22a, b, c, d)$$

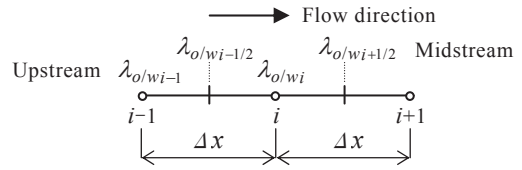


Fig.6 Evaluation points of the mobility factor.

Equation (20) can be solved implicitly associated with the boundary conditions of (18a,b) using well-known tridiagonal matrix solver⁶. Once pressure is computed, water saturation can be calculated explicitly as follows.

$$\begin{aligned} S_{wi}^{k+1} &= \frac{\gamma}{\phi} \{ \lambda_{wi} p_{i+1}^{k+1/2} - (\lambda_{wi} + \lambda_{wi-1}) p_i^{k+1/2} \\ &\quad + \lambda_{wi-1} p_{i-1}^{k+1/2} \} + S_{wi}^k \end{aligned} \quad (23)$$

Owing to high nonlinearities in the pressure and saturation equations, calculations of Eqs.(20)(23) must be iterated until successive changes of $p_i^{k+1/2}$ and S_{wi}^{k+1} are settled in prescribed tolerances. Picard method and/or Newton-Raphson method can be employed to accelerate the convergence⁵.

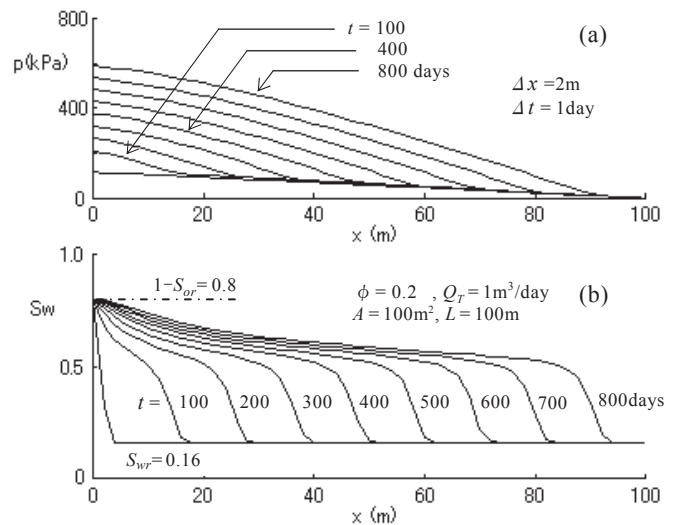


Fig.7 Calculated results of the Buckley-Leverett problem by FDM; (a) Pressure profile, (b) Saturation profile.

In the numerical analysis, it is convenient to make a functional approximation for relative permeabilities, and the following forms of approximation were used in the simulation.

$$k_{rw} = 0.23S_e^{5.5} + 0.18S_e \quad (24a)$$

$$k_{ro} = 0.86(1-S_e)^{4.8} + 0.04(1-S_e) \quad (24b)$$

where S_e is the effective saturation defined by

$$S_e = \frac{S_w - S_{wr}}{1 - S_{wr} - S_{or}} \quad (24c)$$

The calculated results by FDM under the same reservoir condition and physical parameters are shown in Fig.7. The grid spacing and time step size selected were $\Delta x=2m$ and $\Delta t=1day$. The numerical results showed good coincidence with the results by the Buckley-Leverett analysis previously shown in Fig.5, but a smearing of the displacement front appears. This can be improved by the use of finer grid spacing as illustrated in Fig.8 and Fig.9.

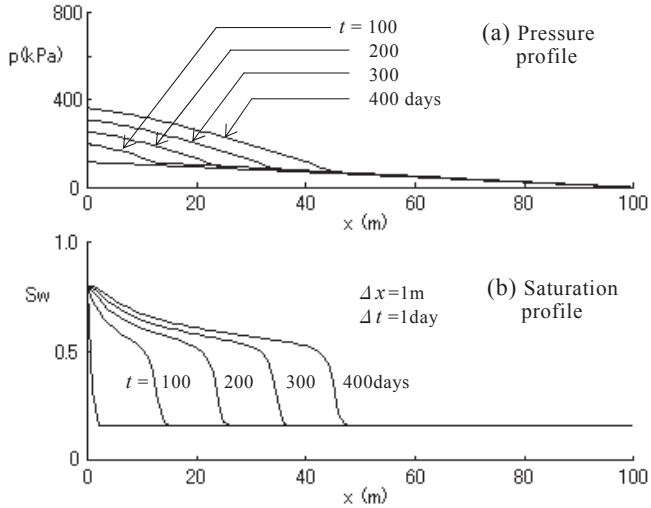


Fig.8 Improvement of the smearing of sharp front with the use of finer grid spacing.

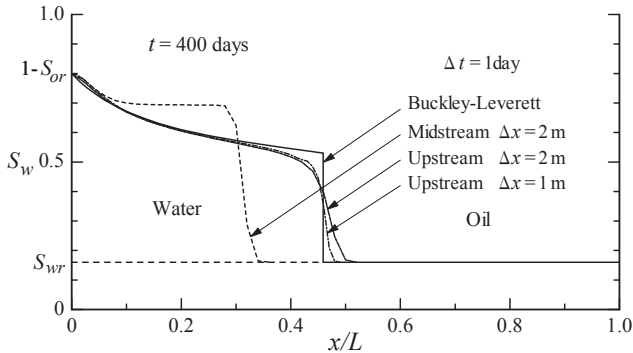


Fig.9 Comparison of saturation front by weighting of mobility factors and grid spacing.

In the FD analysis developed here, the upstream weighting for mobility factors were introduced. If midstream weighting of Eqs.(22a,c) are used, an unrealistic saturation profile comes out as shown in Fig.9. Therefore, the upstream weighting for mobility factors is inevitably required for the two-phase flow simulation by finite difference method.

3.2 Finite Element Method

For the finite element formulation of the boundary value problems, the weighted residual method is frequently employed. Here, the finite element formulation for the Buckley-Leverett problem is performed via Galerkin finite element procedure.

Application of the Galerkin's criterion to Eq.(17) leads to

$$\int_R N_I \left\{ \frac{\partial}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x}) + \frac{\partial}{\partial x} (k_x \lambda_o \frac{\partial p}{\partial x}) \right\} dR = 0 \quad (25)$$

where N_I is the linearly independent weighting function.

Application of the Green's theorem to Eq.(25), it follows

$$\int_R \frac{\partial N_I}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x} + k_x \lambda_o \frac{\partial p}{\partial x}) dR - \int_B N_I q dB = 0 \quad (26)$$

where q is the outward normal flux on the boundary. Introducing the following form of a trial function

$$p(x,t) = N_J(x) p_J(t) \quad (27)$$

and subdividing the region R into m finite elements, Eq.(26) is written in matrix form as

$$\sum_{i=1}^m [E]^e \{p\}^{k+1/2} = \sum_{i=1}^m \{R\}^e \quad (28)$$

in which the elements $[E]^e$ and $\{R\}^e$ are given by

$$E_{IJ}^e = \int_{R^e} k_x (\lambda_w + \lambda_o) \frac{\partial N_I}{\partial x} \frac{\partial N_J}{\partial x} dR \quad (29a)$$

$$R_I^e = \int_{B^e} N_I [(q^{k+1} + q^k) / 2] dB \quad (29b)$$

In the similar manner, finite element formulation for Eq.(15) is performed via Galerkin method.

$$\int_R N_I \left\{ \frac{\partial}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x}) - \phi \frac{\partial S_w}{\partial t} \right\} dR = 0 \quad (30)$$

By Green's theorem

$$\int_R \frac{\partial N_I}{\partial x} (k_x \lambda_w \frac{\partial p}{\partial x}) dR - \int_B N_I q dB + \int_R N_I \phi \frac{\partial S_w}{\partial t} dR = 0 \quad (31)$$

Introducing the following form of trial functions

$$p(x,t) = N_J(x) p_J(t) \quad , \quad S_w(x,t) = N_J(x) S_{wJ}(t) \quad (32a,b)$$

and subdividing the region R into m finite elements, Eq.(30) is written in matrix form as

$$\sum_{i=1}^m [F]^e \{S_w\}^{k+1} = \sum_{i=1}^m \{Q\}^e \quad (33)$$

in which the elements $[F]^e$ and $\{Q\}^e$ are given by

$$F_{IJ}^e = \int_{R^e} \frac{\phi}{\Delta t} N_I N_J dR \quad (34a)$$

$$Q_I^e = \left(\int_{R^e} \frac{\phi}{\Delta t} N_I N_J dR \right) S_{wJ}^k - \left(\int_{R^e} k_x \lambda_w \frac{\partial N_I}{\partial x} \frac{\partial N_J}{\partial x} dR \right) p_J^{k+1/2} + \int_{B^e} N_I q^{k+1/2} dB \quad (34b)$$

Element matrix of Eq.(34a) is the capacitance matrix, and integral on the second term of Eq.(34b) is the conductance matrix. Experience indicates that the capacitance matrix should be lumped by diagonalizing the element components to eliminate numerical oscillation. The upstream weighting for mobility factors was also introduced. The matrix equations (28) and (33) are solved using an appropriate solver until successive changes of $p_i^{k+1/2}$ and S_{wi}^{k+1} are settled in the prescribed tolerances.

Figure 10 illustrates the calculated results by FEM up to $t = 400$ days. Pressure and saturation profiles are the same as FD solutions previously shown in Fig.7.

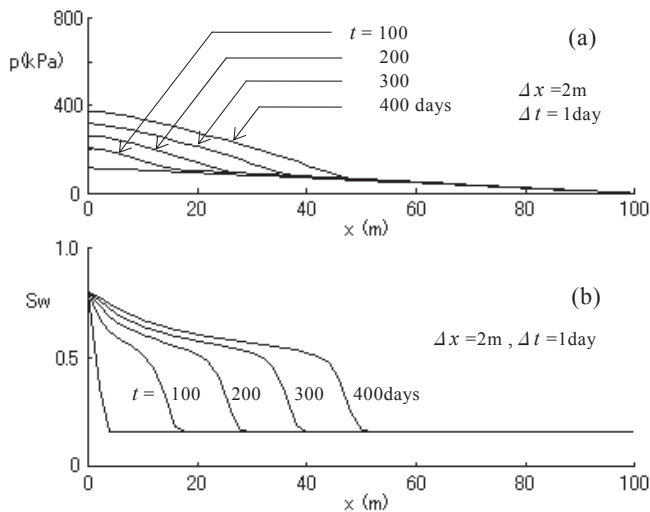


Fig.10 Calculated results of the Buckley-Leverett problem by FEM; (a) Pressure profile, (b) Saturation profile.

Through the numerical simulations by finite difference and finite element methods for oil displacement phenomena, we can obtain not only the saturation profile but also the pressure profile in the reservoir, which provide us useful information for the waterflooding operation in the petroleum production technology.

3. Conclusion

A quantitative analysis for the displacement of oil by water

was performed based on the Buckley-Leverett frontal displacement theory and the numerical methods. The major conclusions obtained through this study are as follows:

- 1) The Buckley-Leverett equation is a first-order hyperbolic equation with respect to saturation change, and which exhibits highly convection-dominated situation. The advance of saturation front is controlled by the derivative of fractional flow rates to water saturation. The quantitative demonstration revealed a constant progress of abrupt leading edge for the saturation front.
- 2) The simultaneous two-phase flow equations can be transformed to a pair of pressure equation and saturation equation, under the assumption of neglecting capillary pressure. The equations can be solved successively with IMPES (Implicit Pressure-Explicit Saturation) scheme by finite difference method with an aid of upstream weighting for the mobility factors.
- 3) The Buckley-Leverett problem can be solved by finite element method using Galerkin finite element formulation of pressure equation and saturation equation subsequently. Upstream weighting is also required, and the results showed good agreement with the Buckley-Leverett solution.

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